SECTIONS OF LIE GROUP ACTIONS AND A THEOREM BY M. NEWMAN

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Abstract. Let M be a smooth finite-dimensional manifold, G a Lie group, and $\Phi : G \times M \to M$ a smooth action. Consider the following mapping

$$\varphi : C^{\infty}(M, G) \to C^{\infty}(M, M),$$

defined by $\varphi(\alpha)(x) = \alpha(x) \cdot x$, for $\alpha \in C^{\infty}(M, G)$ and $x \in M$. In this paper we describe the structure of inverse images of elements of $C^{\infty}(M, M)$ under φ for $\dim G = 1$, i.e. when G is either $\mathbb{R}$ or S^1. As an application we obtain a new proof of the well-known theorem by M. Newman concerning the interior of the fixed point set of a Lie group action.

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1. Introduction

Let M be a smooth (C^{∞}) connected finite-dimensional manifold, G a Lie group and

$$\Phi : G \times M \to M \tag{1.1}$$

a smooth action of G on M. We will often write $g \cdot z$ instead $\Phi(g, z)$ for $g \in G$ and $z \in M$.

For each point $z \in M$ let $\mathcal{O}_z \subset M$ be the corresponding orbit of z. Let also $\mathrm{Fix}\,\Phi$ designates the fixed-point set of Φ. A point which is not fixed will be called *regular*.

Suppose that a mapping $f : M \to M$ preserves each orbit of Φ, i.e. $f(\mathcal{O}_z) \subset \mathcal{O}_z$ for all $z \in M$. The following question arises often in different problems: can f be "smoothly parametrized" by elements of G? Thus we need a smooth mapping $\alpha : M \to G$ such that

$$f(z) = \Phi(\alpha(z), z) = \alpha(z) \cdot z,$$

for all $z \in M$? This question leads to the consideration of the following mapping:

$$\varphi : C^{\infty}(M, G) \to C^{\infty}(M, M), \tag{1.2}$$

defined by

$$\varphi(\alpha)(z) = \Phi(\alpha(z), z) = \alpha(z) \cdot z,$$

for $\alpha \in C^{\infty}(M, G)$ and $z \in M$.

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Evidently, the identity mapping id_M of M belongs to the image of φ. Consider the inverse image $Z_{\mathrm{id}} = \Phi^{-1}(\mathrm{id}_M) \subset C^\infty(M,G)$. This set of smooth mappings $M \to G$ plays an important role for the understanding the structure of φ. It is easy to see that Z_{id} is a subgroup of the $C^\infty(M,G)$. Moreover, we will show that the inverse images of mappings $C^\infty(M,M)$ of φ coincide with the adjacent classes of $C^\infty(M,G)$ by Z_{id}.

The main result of this paper is a full description of the group Z_{id} for the case $\dim G = 1$, i.e. when G is either $\mathbb{R}^1$ or S^1.

For simplicity let us denote by F the fixed-point set $\mathrm{Fix}\,\Phi$ of the action (1.1). Let also $\mathrm{Int}\,F$ and $\mathrm{Fr}\,(\mathrm{Int}\,F)$ be respectively the interior of F in M and the frame of this interior.

1.1. **Theorem.** *Suppose that $G = \mathbb{R}$, i.e. Φ is a flow on M.*

(1) *If $\mathrm{Int}\,F \neq \emptyset$, then*

$$Z_{\mathrm{id}} = \{\mu \in C^\infty(M,\mathbb{R}) \mid \mu|_{M\setminus \mathrm{Int}\,F} = 0\}.$$

(2) *Suppose that $\mathrm{Int}\,F = \emptyset$. Then we have two possibility: either $Z_{\mathrm{id}} = \{0\}$ or there exists a smooth strictly positive smooth function $\mu : M \to (0,\infty)$, such that*

$$Z_{\mathrm{id}} = \{n\mu \mid n \in \mathbb{Z}\} \approx \mathbb{Z}.$$

In this case each regular point z of Φ is periodic and $\mu(z)$ is equal to the period of z.

1.2. **Theorem.** *Suppose $G = S^1$. Let K be the ineffectivity of the action (1.1), i.e. the kernel of the induced homomorphism $G \to \mathrm{Diff}\,M$. If the action Φ is non-trivial, then $\mathrm{Int}\,F = \emptyset$ and Z_{id} consists of constant mappings $M \to K \subset G$. Consequently, Z_{id} is isomorphic to K and is a finite cyclic group.*

As an application we obtain a new proof of the following theorem 1.3. It is a variant of the well-known theorem by M. Newman [N31] concerning the actions of cyclic groups, see also [M57, MSZ] for more general account.

1.3. **Theorem.** *Let G be a compact Lie group acting non-trivially on a finite-dimensional manifold. Then the set of fixed points of G is nowhere dense.*

Proof. Since G is compact, it contains at least one one-parametric subgroup isomorphic to $SO(2) \approx S^1$. Indeed, let G_1 be arbitrary 1-parametric subgroup in G. Then its closure $\overline{G_1}$ is a compact abelian Lie group, i.e. a torus, and therefore has S^1-subgroups.

Let $\mathrm{Fix}\,S$ be the fixed-point set of the induced action of S on M. Then $\mathrm{Fix}\,G \subset \mathrm{Fix}\,S$. By Theorem 1.2, $\mathrm{Fix}\,S$ is nowhere dense, therefore so is $\mathrm{Fix}\,G$. □

1.4. **Structure of the paper.** In §2 we consider the general properties of the mapping φ that hold for arbitrary Lie groups G. Further, we shall confine ourselves with the case $G = \mathbb{R}$ only. In §3 we consider the behavior of functions belonging to Z_{id} near regular points of Φ. §4 includes a lemma about the lower bound of the periods of trajectories of linear flows. This lemma will be used in §5 where we prove two statements about local behavior of functions from Z_{id}. Finally, in §6 and §7 we prove Theorems 1.1and 1.2.

2. Properties of φ.

2.1. **Lemma.** *The image of* $\operatorname{im}\varphi$ *is a subsemigroup in* $C^\infty(M,M)$. *Moreover, the intersection* $\operatorname{im}\varphi \cap \operatorname{Diff} M$ *is a subgroup of* $\operatorname{Diff} M$.

Proof. Let $\alpha,\beta,\gamma \in C^\infty(M,G)$ and let $f,g,h \in C^\infty(M,M)$ be respectively their images under φ. Suppose also that h is a diffeomorphism. For the proof of lemma we have to find mappings

$$\sigma_{f\circ g}, \sigma_{h^{-1}} : M \to G,$$

such that $f\circ g = \varphi(\sigma_{f\circ g})$ and $h^{-1} = \varphi(\sigma_{h^{-1}})$. It is not easy to see that the following mappings satisfy the conditions above:

$$\sigma_{f\circ g}(z) = \alpha(g(z))\cdot\beta(z), \tag{2.3}$$

$$\sigma_{h^{-1}}(z) = \left(\gamma\left(h^{-1}(z)\right)\right)^{-1}, \tag{2.4}$$

for all $z\in M$. Indeed,

$$f\circ g(z) = \alpha(g(z))\cdot g(z) = \alpha(g(z))\cdot\beta(z)\cdot z = \sigma_{f\circ g}(z)\cdot z.$$

For the proof of (2.4) notice that the identity $h\left(h^{-1}(z)\right) = z$ means that $\gamma\left(h^{-1}(z)\right)\cdot h^{-1}(z) = z$, whence,

$$h^{-1}(z) = \gamma\left(h^{-1}(z)\right)^{-1}\cdot z = \sigma_{h^{-1}}(z)\cdot z. \quad \square$$

2.2. Denote by $Z_{\mathrm{id}}(\Phi)$ the inverse image of the identity mapping: $\varphi^{-1}(\mathrm{id}_M) \subset C^\infty(M,G)$

$$Z_{\mathrm{id}}(\Phi) := \varphi^{-1}(\mathrm{id}_M). \tag{2.5}$$

Then $\mu(z)\cdot z = z$ for all $\mu\in Z_{\mathrm{id}}(\Phi)$ $z\in M$.

The following statement is evident.

2.3. **Proposition.** *The set* $Z_{\mathrm{id}} = Z_{\mathrm{id}}(\Phi)$ *has the following properties:*

1. Z_{id} *is a subgroup in*$C^\infty(M,G)$.
2. *Let* $\alpha,\beta\in C^\infty(M,G)$. *Then* $\varphi(\alpha)=\varphi(\beta)$ *if and only if* $\alpha^{-1}\cdot\beta\in Z_{\mathrm{id}}$. $\square$

2.4. **Lemma.** *Suppose that* $\operatorname{Int} F \neq \emptyset$ *and let* $\alpha,\beta\in C^\infty(M,G)$ *coincide outside the interior of* $\operatorname{Int} F$*:*

$$\alpha|_{M\setminus \operatorname{Int} F} = \beta|_{M\setminus\operatorname{Int} F}.$$

Then $\varphi(\alpha)=\varphi(\beta)$. *In particular, if* $\alpha(z) = e\in G$ *for all* $z\in M\setminus\operatorname{Int} F$, *then* $\alpha\in Z_{\mathrm{id}}$.

Proof. We have to prove that $\alpha(z)\cdot z = \beta(z)\cdot z$ for $z\in M$. Let $z\in M\setminus\operatorname{Int} F$. Then $\alpha(z)=\beta(z)$, whence $\alpha(z)\cdot z = \beta(z)\cdot z$.

Suppose that $z\in\operatorname{Int} F$. Then $t\cdot z = z$ for each $t\in G$, whence $\alpha(z)\cdot z = \beta(z)\cdot z = z$. $\square$

The following lemma is also evident.

2.5. **Lemma.** *A constant mapping* $\mu : M\to G$ *belongs to* Z_{id} *if and only if the image of* μ *belongs to the ineffectivity kernel of the action* Φ. $\square$

3. Regular points of flows

In the sequel, it is assumed that $G = \mathbb{R}$. It is also convenient to assume that Φ is a *local* action. Thus Φ is a local flow defined on some open subset of M. Let us recall the definitions.

3.1**. Definition.** Let U be an open connected subset of M and $\mathcal{J}$ an open interval in $\mathbb{R}$ containing 0. A smooth mapping

$$\Phi : \mathcal{J} \times U \to M \tag{3.6}$$

is a *local flow* if the following conditions hold true:

(1) $\Phi(0, x) = x, \forall x \in U$,
(2) $\Phi(s, \Phi(t, x)) = \Phi(t + s, x)$, for $x \in U$ and $t, s \in \mathcal{J}$ provided $\Phi(t, x) \in U$ and $t + s \in \mathcal{J}$.

In the case $U = M$ and $\mathcal{J} = \mathbb{R}$ the flow Φ is *global.* For each $t \in \mathcal{J}$ the restriction of Φ to $\{t\} \times U$

$$\Phi|_{\{t\}\times U} : U \to M$$

will be denoted by Φ_t.

Let $z \in M$. The *orbit* of z is the following set $\Phi(\mathcal{J} \times \{x\}) \subset M$. A point $z \in U$ is *fixed* with respect to the flow provided $\Phi(t, z) = z$ for each $t \in \mathcal{J}$. All other points are *regular*. A regular point z is *periodical* if $\Phi(t, z) = z$ for some $t > 0$. The least such number t is called the *period* of z and is denoted by $\mathrm{Per}\,(z)$. The orbit of a periodic point is *closed*, and the orbit of a regular but non-periodic is *non-closed.*

3.2. Notice that a local flow (3.6) induces the following mapping

$$\varphi : C^\infty(U, \mathcal{J}) \to C^\infty(U, M) \tag{3.7}$$

defined by $\varphi(\alpha)(z) = \Phi(\alpha(z), z)$, for $z \in U$ and $\alpha \in C^\infty(U, \mathcal{J})$.

Let $\mathrm{id}_U : U \to M$ be the identity embedding and designate

$$Z_{\mathrm{id}} := \varphi^{-1}(\mathrm{id}_U).$$

3.3**. Lemma.** *Let ω be a non-constant orbit of Φ and $\mu \in Z_{\mathrm{id}}$. If ω is non-closed, then $\mu|_\omega = 0$. If ω is closed orbit of period θ, then $\mu|_\omega = n\theta$ for some $n \in \mathbb{Z}$.*

Proof. Recall that the condition $\mu \in Z_{\mathrm{id}}$ means that $\Phi(\mu(x), x) = x$ for all $x \in M$.

Consider a non-closed orbit ω of Φ. Then for arbitrary pair $x, y \in \omega$ there exists a *unique* number $t \in \mathcal{J}$ such that $\Phi(t, x) = y$. In particular, $t = 0$ iff $x = y$. Therefore $\mu(x) = 0$ for all $x \in \omega$.

Let ω be a non-closed orbit of period θ and $x \in \omega$. Then $\Phi(t, x) = x$ if and only if $t = n\theta$ for some $n \in \mathbb{Z}$. Therefore $\mu(x) = n(x)\theta$ for all $x \in \omega$, where $n = \mu/\theta : \omega \to \mathbb{Z}$ is a *continuous* function. It follows that n is constant, i.e. $\mu|_\omega = n\theta$. □

The following lemma claims a local uniqueness of function of Z_{id} in a neighborhood of a regular point of a flow.

3.4. **Lemma.** *Let C be a connected component of the set of regular points of Φ. Let also $\alpha, \beta \in C^\infty(U, \mathcal{J})$ be such that $\varphi(\alpha) = \varphi(\beta)$. If $\alpha(y) = \beta(y)$ for some $y \in C$, then $\alpha|_C = \beta|_C$. In particular, if $\alpha \in Z_{\mathrm{id}}$ and $\alpha(y) = 0$, then $\alpha|_C = 0$.*

Proof. It suffices to show that $\alpha = \beta$ in a neighborhood of $y \in C$. We will now express a function α through $\varphi(\alpha)$ in a neighborhood of a regular point of Φ.

Let $f = \varphi(\alpha)$ and $a = \alpha(y)$. Since the point $z = f(y) = \Phi_a(y)$ is also regular, there are local coordinates $(x_1, \dots, x_n)$ in a some neighborhood W of z such that $z = 0$ and $\Phi(t, x_1, \dots, x_n) = (x_1 + t, x_2, \dots, x_n)$. Consider the following neighborhood $V = f^{-1}(W) \cap \Phi_{-a}(W)$ of y. Evidently,

$$\alpha(x) = p_1 \circ \varphi(\alpha) \circ \Phi(a, x) - p_1 \circ \Phi(a, x), \quad \forall x \in V. \tag{3.8}$$

where $p_1 : \mathbb{R}^n \to \mathbb{R}$ is a projection onto the first coordinate.

This formula proves our lemma. Indeed, if $\varphi(\alpha) = \varphi(\beta)$ and $\alpha(y) = \beta(y) = a$, then by formula (3.8) $\alpha \equiv \beta$ in some neighborhood of y consisting of regular points only. Therefore they coincides on C. □

4. Periods of linear flows

We prove here a lemma that gives a lower bounds for the periods of orbits of linear flows. First we introduce some notation and recall the "real" Jordan form of a matrix.

Let A be a $(k \times k)$-matrix. Then a *Jordan cell* $J_p(A)$ of A is the following $(pk \times pk)$-matrix:

$$J_p(A) = \left.\begin{Vmatrix} A & 0 & \cdots & 0 & 0 \\ E_k & A & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & A & 0 \\ 0 & 0 & \cdots & E_k & A \end{Vmatrix}\right\} p,$$

where E_k is a unit $(k \times k)$-matrix. For $\alpha, \beta \in \mathbb{R}$ set

$$R\,(\alpha + i\beta) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{4.9}$$

Let X be a square matrix with real entries and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ an eigen value of X. Then $\overline{\lambda}$ is also an eigen value of X. Let $\lambda_1, \overline{\lambda_1}, \dots, \lambda_c, \overline{\lambda_c}$ be all complex and $\lambda_{c+1}, \dots, \lambda_{c+d}$ all real eigen values of X. Then by Jordan theorem X is conjugated to the following matrix:

$$\operatorname{diag}\,[J_{p_1}(R\,(\lambda_1)), \dots, J_{p_s}(R\,(\lambda_c)), J_{k_1}(\lambda_{c+1}), \dots, J_{k_m}(\lambda_{c+d})]. \tag{4.10}$$

(see e.g. Theorem 2.2.5 in [PM].)

We will also need the following formula for the exponent of a Jordan cell:

$$e^{J_k(A)t} = \begin{Vmatrix} e^{At} & 0 & \cdots & 0 \\ t \cdot e^{At} & e^{At} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{t^{k-1}}{(k-1)!} \cdot e^{At} & \frac{t^{k-2}}{(k-2)!} \cdot e^{At} & \cdots & e^{At} \end{Vmatrix}. \tag{4.11}$$

Let $M(n)$ be the algebra of all real $n \times n$-matrices and $\exp : M(n) \to GL(\mathbb{R}, n)$ an exponential mapping.

4.1. **Lemma.** *Let $A \in M(n)$ and let $\Lambda = \{\lambda_j\}_{j=1}^{r}$ be the set of its purely imagine non-zero eigen values. Then the linear flow $\Phi(t,x) = e^{At}x$ has a closed orbit if and only if $\Lambda \neq \emptyset$. In this case the period of each closed orbit of Φ is $\geq \min\limits_{j=1..r} \frac{2\pi}{|\lambda_j|}$.*

Proof. First suppose that A is a Jordan cell. Then Φ has closed orbits iff $A = J^p(R\,(i\beta))$ for some $\beta \in \mathbb{R} \setminus \{0\}$. In this case all eigen values of A are equal to $\pm i\beta$. Moreover, it follows from formula (4.11) that all closed orbits of Φ belong to the invariant subspace generated by two latter coordinates. They also have the same period $\frac{2\pi}{|\beta|} = \frac{2\pi}{|\lambda|}$.

Consider now the general case. We can assume that A has a real normal Jordan form (4.10), and that for $i = 1..r$ ($r \leq c$) the eigen values λ_i constitute Λ. Set $m = c + d$ and designate by $V_i \subset \mathbb{R}^n (i = 1..m)$ the invariant subspace of Φ corresponding to the corresponding cell either $J_{p_i}(\lambda_i)$ or $J_{p_i}(R\,(\lambda_i))$. Then $\mathbb{R}^n = \mathop{\oplus}\limits_{i=1}^{m} V_i$.

Let $p_i : \mathbb{R}^n \to V_i$ be a natural projection and $\Phi^i = \Phi|_{V_i}$ the restriction of Φ to V_i. Then for every orbit ω of Φ and every $i = 1..m$ the set $\omega_i = p_i(\omega)$ is the orbit of the flow Φ^i. Moreover, ω is closed iff all ω_i are either closed of constant and for at least one $j = 1..r$ the ω_j is closed. Therefore, Φ has a closed orbit iff $\Lambda \neq \emptyset$.

Suppose that ω is a closed orbit of Φ of a period θ. Let $\omega_j = p_j(\omega)$ be the projection of ω being a closed orbit of some Φ^j. Then the period of ω_j is equal to $\theta_j = \frac{2\pi}{|\lambda_j|}$. Since the projection p_j factors Φ onto Φ^j, i.e. $p_j \circ \Phi_t = \Phi^j_t \circ p_j$, it follows that $\theta = s\theta_j$ for some $s \in \mathbb{N}$. In particular, $\theta \geq \theta_j \geq \min\limits_{k=1..r} \frac{2\pi}{|\lambda_k|}$. □

4.2. **Corollary.** *Let $\{A_i\}_{i\in\mathbb{N}} \subset M(n)$ be a sequence of matrices such that for all $i \in \mathbb{N}$ the linear flow $\Phi_i(t,x) = e^{A_i t}x$ has closed orbits. Let θ_i be the minimum of periods of orbits of Φ_i. If $\lim\limits_{i\to\infty} A_i = 0$, then $\lim\limits_{i\to\infty} \theta_i = \infty$.*

Proof. Let Λ_i be the set of those eigen values of A_i that correspond to closed orbits of Φ_i (see Lemma 4.1.) Then $\Lambda_i \neq \emptyset$ for all $i \in \mathbb{N}$. Set $\widetilde{\lambda}_i = \max\limits_{\lambda\in\Lambda_i} |\lambda|$. Then $\theta_i = 2\pi/\widetilde{\lambda}_i$. Since $\lim\limits_{i\to\infty} A_i = 0$, it follows from the continuity of a spectrum of matrices that $\lim\limits_{i\to\infty} \widetilde{\lambda}_i = 0$. Therefore by Lemma 4.1, $\lim\limits_{i\to\infty} \theta_i = \lim\limits_{i\to\infty} 2\pi/\widetilde{\lambda}_i = \infty$. □

5. Fixed and periodic poitns of flows

5.1. **Proposition.** *Let $R = U \setminus F$ be the set of regular points of a flow Φ, V be a connected component of R, and $z \in F \cap \overline{V}$. Let also $\mu \in Z_{\mathrm{id}}$ be such that $\mu(z) = 0$. Then $\mu \equiv 0$ on $\overline{V}$.*

Proof. Notice that the components of R can be divided into the following two parts

$$R = N \cup P,$$

where N consists of those components that include at least one non-closed orbit of Φ and P consists of components with only periodic orbits.

Let $\mu \in Z_{\mathrm{id}}$. Then it follows from Lemmas 3.3 and 3.4 that $\mu|_{\overline{N}} = 0$. Thus it suffices to prove our proposition for the component belonging to $V \subset P$.

Let $\{z_i\}_{i\in\mathbb{N}} \subset V$ be a sequence of periodic points of Φ converging to z. For each $i \in \mathbb{N}$ let θ_i be the period of z_i. Then by Lemma 3.3, $\mu(z_i) = n_i\theta_i$ for some $n_i \in \mathbb{Z}$. Hence by continuity of μ we get

$$\mu(z_i) = n_i\theta_i \to \mu(z) = 0. \tag{5.12}$$

Taking if necessary a subsequence we can assume that there exists a certain finite or infinite limit $\theta = \lim\limits_{i\to\infty} \theta_i \geq 0$. By Corollary 4.2 we have that $\theta > 0$. Then it follows from (5.12) that $n_i = 0$ for all sufficiently large $i \in \mathbb{N}$. In particular, $\mu(z_i) = 0$ for some $i \in \mathbb{N}$. Since $z_i \in V_\lambda$, we get from Lemma 3.4 that $\mu \equiv 0$ on V_λ. Proposition is proved. □

5.2. **Proposition.** *Let* $\mathrm{Fr}\,(F) = F \setminus \mathrm{Int}\, F$ *be the boundary of the fixed-point set of* Φ*. Suppose that for a point* $z \in \mathrm{Fr}\,(F)$ *one of the following conditions holds true:*

(1) *z belongs to the boundary of the interior of F, i.e.* $z \in \mathrm{Fr}\,(\mathrm{Int}\, F)$*;*
(2) *the tangent linear flow at z is trivial, i.e.* $\frac{\partial\Phi}{\partial x}(t,z) = E_n$ *for all* $t \in \mathcal{J}$*.*

Then for each $\mu \in Z_{\mathrm{id}}$ *we have that* $\mu \equiv 0$ *in some neighborhood of z in the set* $U \setminus \mathrm{Int}\, F$*.*

Proof. Since the problem is local, we can assume that $M = \mathbb{R}^m$ and $z = 0 \in \mathbb{R}^m$ is the origin.

Define the following mapping $\Psi : \mathcal{J}\times U \to GL(\mathbb{R}, n)$ by $\Psi(t,x) = \frac{\partial\Phi}{\partial x}(t,x)$. Since $\Psi(0,x) = E_n$ for all $x \in U$, we see that the mapping $\nu = \exp^{-1} \circ \Psi : \mathcal{J} \times U \to M(n)$ is defined in some neighborhood of $(0,z)$ in $\mathcal{J} \times U$. Thus $\Psi(t,x) = e^{\nu(t,x)}$.

Notice that for each $x \in U$ the restriction $\Psi(*,x) : \mathcal{J} \to GL(\mathbb{R}, n)$ is a local *homomorphism*. Therefore it induces a linear flow on $\mathbb{R}^m$. Hence the matrix $A(x,t) = \nu(t,x)/t$ does not depend on $t \in \mathcal{J}$, i.e. $\Psi(t,x) = e^{A(x)t}$.

Moreover, for each periodic point x the flow $\Psi(*,x)$ has closed trajectories. Indeed, let $\mathcal{F}(x) = \frac{\partial\Phi}{\partial t}(0,x)$ be a vector field generating Φ. Applying to both parts of the following relation

$$\Phi(s,\Phi(t,x)) = \Phi(t,\Phi(s,x))$$

the operator $\frac{\partial}{\partial t}$ and then set $s = 0$ we will obtain

$$\frac{\partial\Phi}{\partial t}(0,\Phi(t,x)) = \frac{\partial\Phi}{\partial x}(t,x)\frac{\partial\Phi}{\partial t}(0,x),$$

whence $\mathcal{F}(\Phi(t,x)) = \Psi(t,x)\mathcal{F}(x)$. This means that the vectors $\mathcal{F}(\Phi(t,x))$ and $\mathcal{F}(x)$ belong to the same orbit of the flow $\Psi(*,x)$. It follows that if x is a periodic point of Φ, then $\mathcal{F}(x)$ is a periodic point of $\Psi(*,x)$ of the same period: $\mathrm{Per}\,(x) \geq \mathrm{Per}\,(\mathcal{F}(x))$.

Now we can complete the proposition. Evidently, (2) holds for each internal point of F. Therefore it also holds for the boundary points of $\mathrm{Fr}\,(\mathrm{Int}\, F)$. Hence (1) implies (2).

Thus suppose that (2) holds true. Consider the following sequences of vectors

$$F_i(t) = \frac{\partial \Phi}{\partial t}(t, z_i)$$

and matrices

$$A_i(t) = \frac{\partial \Phi}{\partial x}(t, z_i)$$

depending on a parameter $t \in \mathcal{J}$.

As noted above, since z_i is a periodic point for Φ, it follows that each vector $F_i(t)$ is also periodic of the same period $\leq \operatorname{Per}(z_i) = \theta_i$. By (2) $\lim\limits_{i\to\infty} A_i(t) = E_n$. Then from Corollary 4.2 we get

$$\theta = \lim_{i\to\infty} \theta_i \geq \lim_{i\to\infty} \operatorname{Per}(F_i(t)) = \infty.$$

Since the value $\mu(z) = \lim\limits_{i\to\infty} n_i\theta_i$ is finite, we see that $\lim\limits_{i\to\infty} n_i = 0$. Hence $\mu(z) = 0$. □

6. Proof of Theorem 1.1

Case (1). Suppose that $\operatorname{Int} F \neq \emptyset$. Denote

$$Z'_{\mathrm{id}} := \{\mu \in C^\infty(M, \mathbb{R}) \mid \mu|_{M\setminus \operatorname{Int} F} = 0\}.$$

We should show that $Z'_{\mathrm{id}} = Z_{\mathrm{id}}$. By Lemma 2.4 $Z'_{\mathrm{id}} \subset Z_{\mathrm{id}}$.

Let $\mu \in Z_{\mathrm{id}}$. By statement (1) of Proposition 5.2 we have that $\mu(z) = 0$ for each $z \in \operatorname{Fr}(\operatorname{Int} F)$. Then by Proposition 5.1 $\mu = 0$ on each connected component $M \setminus \operatorname{Int} F$ containing z.

Since M is connected, it follows that each connected component of the set $M \setminus \operatorname{Int} F$ intersects $\operatorname{Fr}(\operatorname{Int} F)$. Therefore $\mu = 0$ on $M \setminus \operatorname{Int} F$, i.e. $\mu \in Z'_{\mathrm{id}}$. Hence $Z_{\mathrm{id}} \subset Z'_{\mathrm{id}}$. This proves (1).

Case (2). Let $\operatorname{Int} F = \emptyset$. Suppose that $Z_{\mathrm{id}} \neq \{0\}$. We should prove that there exists a smooth function $\mu > 0$ such that $Z_{\mathrm{id}} = \{n \cdot \mu\}_{n\in\mathbb{Z}}$.

Consider at first arbitrary $\mu \in Z_{\mathrm{id}}$. If $\mu(z) = 0$ for at least one point $z \in M$, then by Proposition 5.1 $\mu \equiv 0$ on $M \setminus \operatorname{Int} F = M$. Thus if $\mu(z) \neq 0$, then $\mu \neq 0$ on M. Therefore we can assume that $\mu > 0$ on all of M.

For each point $z \in M$ define the following mapping $\tau_z : Z_{\mathrm{id}} \to \mathbb{R}$ by $\tau_z(\nu) = \nu(z)$. It is easy to see that τ_z is a *homomorphism.* By Proposition 5.1 its kernel is trivial $\ker \tau_z = 0$, i.e. τ_z is injective.

Let z be a regular point of Φ. Then by Lemma 3.3 we see that $\operatorname{im} \tau_z$ is a closed subgroup in $\mathbb{R}$. Hence $\operatorname{im} \tau_z$ is either trivial or isomorphic with $\mathbb{Z}$.

Suppose that $\operatorname{im} \tau_z \approx \mathbb{Z}$. Let $r \in \operatorname{im} \tau_z \subset \mathbb{R}$ be a positive generator of $\operatorname{im} \tau_z$. Then the function $\mu = \tau_z^{-1}(r)$ is a strictly positive generator of the group Z_{id}. This proves (2) and Theorem 1.1. □

6.1. **Example.** Consider the following two flows on the complex plane:

$$\Phi(t, \mathbf{z}) = e^{2\pi i(1+|\mathbf{z}|^2)\cdot t}\mathbf{z}, \qquad \text{and} \qquad \Psi(t, \mathbf{z}) = e^{2\pi i|\mathbf{z}|^2\cdot t}\mathbf{z}.$$

Let us calculate the groups $Z_{\mathrm{id}}(\Phi)$ and $Z_{\mathrm{id}}(\Psi)$.

Evidently, the flows Φ and Ψ have the same trajectories: concentric circles with the center at the origin $0 \in \mathbb{C}$. Nevertheless the corresponding tangent linear flows at 0 of Φ and Ψ differ each from other:

$$\frac{\partial \Phi}{\partial \mathbf{z}}(t,0)\xi = e^{2\pi i t}\xi, \qquad\qquad \frac{\partial \Psi}{\partial \mathbf{z}}(t,0)\xi = \xi,$$

where ξ is a tangent vector at 0. The latter equality means that $\frac{\partial \Psi}{\partial \mathbf{z}}(t,0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then by Propositions 5.1 and 5.2 $Z_{\mathrm{id}}(\Psi) = \{0\}$.

On the other hand, it is easy to see that the following function $\mu(\mathbf{z}) = \frac{1}{1+|\mathbf{z}|^2}$ belongs to $Z_{\mathrm{id}}(\Phi)$. Whence $Z_{\mathrm{id}}(\Phi) \neq \{0\}$. Since the fixed-point set $\mathrm{Fix}\,\Phi = \{0\}$ is nowhere dense in $\mathbb{C}$, it follows from Theorem 1.1that $Z_{\mathrm{id}} \approx \mathbb{Z}$. It is also clear that for each point $\mathbf{z} \neq 0$ the value $\mu(\mathbf{z})$ is equal to the period of $\mathbf{z}$. Then again by Theorem 1.1we see that μ is a generator of the group

$$Z_{\mathrm{id}}(\Phi) = \{n \cdot \mu(\mathbf{z})\}_{n \in \mathbb{Z}}.$$

7. Proof of Theorem 1.2

Let $p : \mathbb{R} \to S^1$ be the universal covering of S^1. Define the following mapping

$$\widetilde{\Phi} : \mathbb{R} \times M \to M$$

by $\widetilde{\Phi}(t,z) = \Phi(p(t),z)$. Then $\widetilde{\Phi}$ is the action of the group $\mathbb{R}$ on M, i.e. a flow, that covers the action Φ. Set

$$\widetilde{Z_{\mathrm{id}}} = Z_{\mathrm{id}}(\widetilde{\Phi}) = \{\tilde{\alpha} \in C^{\infty}(M,\mathbb{R}) \mid \tilde{\alpha}(z) \cdot z = z, \forall z \in M\}.$$

Suppose that $\alpha : M \to S^1$ belongs to Z_{id}, i.e. $\Phi(\alpha(x),x) = x$ for all $x \in M$. Let $z \in M$ be a point and U a small neighborhood of z. Then α lifts to a function $\tilde{\alpha} : U \to \mathbb{R}$ such that $p(\tilde{\alpha}(x)) = \alpha(x)$ for all $x \in U$. Hence

$$\widetilde{\Phi}(\tilde{\alpha}(x),x) = \Phi(p(\tilde{\alpha}(x)),x) = \Phi(\alpha(x),x) = x.$$

This means that $\tilde{\alpha}$ belongs to $\widetilde{Z_{\mathrm{id}}}$ "locally". Notice also that we can always choose $\tilde{\alpha}$ so that $\tilde{\alpha}(z) \neq 0$.

Suppose that $z \in \mathrm{Fr}\,(\mathrm{Int}\,\mathrm{Fix}\,\Phi)$. Then by Theorem 1.1applied to the restriction of Φ to U we get that $\tilde{\alpha}(z) = 0$. This contradicts to the choice of $\tilde{\alpha}$. Hence such a point z does not exists, i.e. $\mathrm{Fr}\,(\mathrm{Int}\,F) = \emptyset$. This is possible only when either $\mathrm{Int}\,F = M$ or $\mathrm{Int}\,F = \emptyset$. In the former case the action $\widetilde{\Phi}$ is trivial, whence $\mathrm{Int}\,F = \emptyset$.

Since $p(\mathbb{Z}) = 1 \in S^1$, we see that the group $\widetilde{Z_{\mathrm{id}}}$ includes all constant functions $M \to \mathbb{Z}$. Therefore is includes more than one element. Whence by (2) of Theorem 1.1 $\widetilde{Z_{\mathrm{id}}} \approx \mathbb{Z}$. Therefore $\widetilde{Z_{\mathrm{id}}}$ consists of constant mappings from M to some subgroup P of $\mathbb{R}$ such that $P \approx \mathbb{Z}$. It follows from Lemma 2.5 that P coincides with the innefectivity kernel $\widetilde{K}$ of the action $\widetilde{\Phi}$.

Therefore the subgroup $p(\widetilde{K}) \subset S^1$ is the ineffectivity kernel K of the action Φ and Z_{id} consists of constant mappings $M \to K$. Moreover, since the kernel of the homomorphism p is infinite, we obtain that $Z_{\mathrm{id}} \approx K \approx \widetilde{K}/\ker p$ is a finite cyclic group.

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